Technical Notes

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Pseudospectral Approximation in a Three-Dimensional Navier-Stokes Code

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Introduction

PSEUDOSPECTRAL methods have the potential of yielding high accuracy for a given number of spatial nodes or collocation points in solving certain classes of partial-differential equations. Spectral and pseudospectral methods are formally infinite-order accurate, according to Gottlieb and Orszag.[1] For problems with periodic boundary conditions, Fourier spectral methods can be efficiently implemented to obtain highly accurate solutions. In this Note we describe a solution scheme for three-dimensional Navier-Stokes equations using a pseudospectral approximation in one of the spatial directions.

Pulliam and Steger[2] developed an implicit finite difference scheme for solving three-dimensional Navier-Stokes equations with a thin-layer approximation in arbitrary curvilinear coordinate systems. It is applied to compute flowfields over missile-shaped configurations at moderate angles of attack. Typically, one of the curvilinear coordinates is chosen to wrap around the body. We use pseudospectral approximations based on Fourier series to compute appropriate derivatives in this circumferential direction and finite difference approximations in the other directions.

Method

The approximate factorization scheme developed by Pulliam and Steger[2] for solving three-dimensional Navier-Stokes equations can be written as

$$\begin{aligned}(I+h\delta_\xi \hat{A}^n-\epsilon_I J^{-1}\nabla_\xi\Delta_\xi J)\,(I+h\delta_\eta \hat{B}^n-\epsilon_I J^{-1}\nabla_\eta\Delta_\eta J)\\ \times(I+h\delta_\zeta \hat{C}^n-hRe^{-1}\delta_\zeta\hat{M}^n-\epsilon_I J^{-1}\nabla_\zeta\Delta_\zeta J)\\ \times(\hat{q}^{n+1}-\hat{q}^n)=\text{R.H.S.}\end{aligned}\tag{1}$$

where

$$\text{R.H.S.}=-\Delta t\cdot\text{RES}-\text{DIS}\tag{2}$$

$$\text{RES}=\delta_\xi\hat{E}^n+\delta_\zeta\hat{G}^n+\delta_\eta\hat{F}^n-Re^{-1}\delta_\zeta\hat{S}^n\tag{3}$$

Here $h=\Delta t$ and δ, Δ, and ∇ are central, forward, and backward difference operators. DIS is a fourth-order numerical dissipation term added to control nonlinear instabilities and RES is the residue term. The terms in the R.H.S. (right-hand side) are approximated by fourth-order accurate differences and the left side operators are approximated by second-order accurate differences. The R.H.S. is driven to zero asymptotically by the left-side operators which reduce to solving a sequence of block-tridiagonal system of equations at each time step. Using a spectral approximation of the η derivatives in the convection terms in the residue RES, it is possible to achieve a particular level of accuracy with fewer circumferential points than is possible with a fourth-order scheme.

We consider the flow over a hemisphere cylinder at an angle of attack. The flow is three dimensional but has a plane of symmetry. In a curvilinear grid, as shown in Pulliam and Steger,[2] the coordinate ξ varies along the body in the streamwise direction, η in the circumferential direction, and ζ away from the body. The velocity components u and w and the coordinates x and z are even functions in η and the velocity component v and the coordinate y are odd functions in η. Hence, the third component of the flux vector $\hat{F}$ is an even function in η and the other components of $\hat{F}$ are odd functions in η.

The components of $\partial\hat{F}/\partial\eta$ are approximated spectrally by even or odd discrete Fourier transforms as follows. Let $f_j=f(\theta_j)$, where $\theta_j=\pi j/N$, $j=0,1,\ldots N$ is a real periodic odd

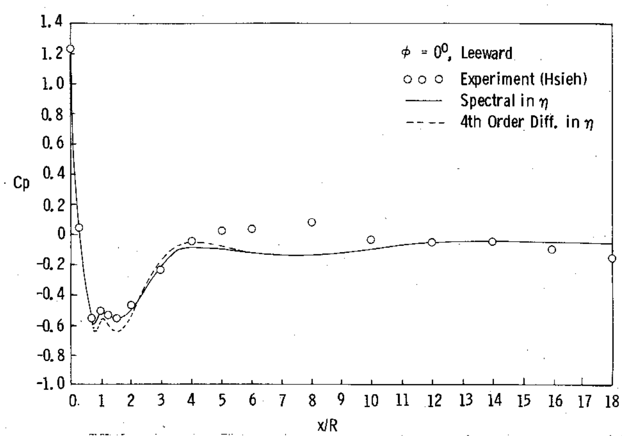

Fig. 1 Hemisphere cylinder at $M_\infty=1.2$, $\alpha=19$ deg, mesh (48×18×30).

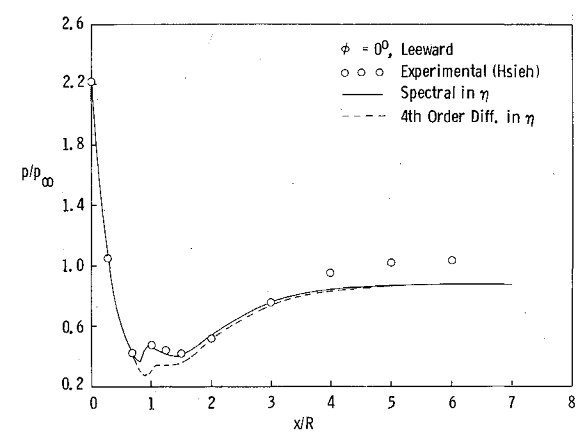

Fig. 2 Hemisphere cylinder at $M_\infty=1.2$, $\alpha=19$ deg, mesh (30×12×30).

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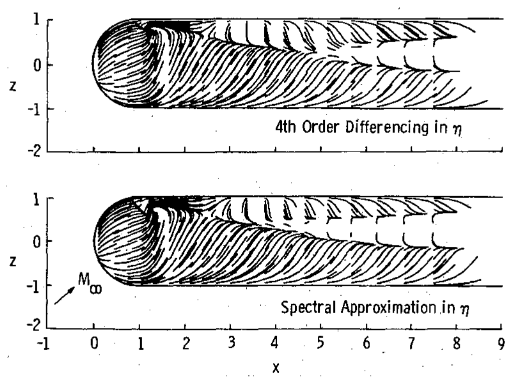

Fig. 3 Streak-line pattern on hemisphere cylinder at $M_\infty = 1.2$, $\alpha = 19$ deg (simulation of oilflow pattern on the surface).

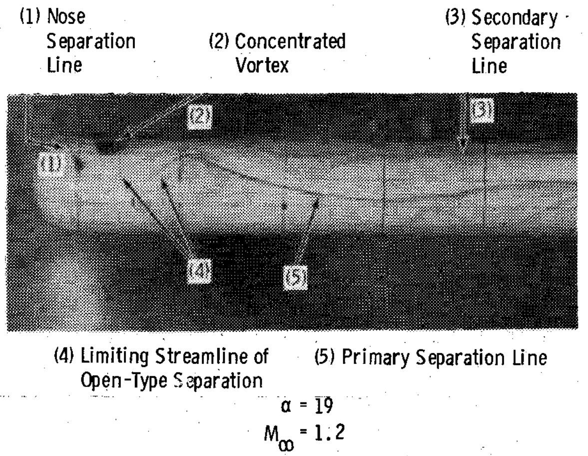

Fig. 4 Oilflow picture showing the surface flow pattern about hemisphere cylinder (from Ref. 4).

discrete function. Then,

$$f_j = -f_{2N-j}, \quad f_j = f_{2N+j} \qquad (f_0 = f_N = 0)$$

and we can express f_j as

$$f_j = \sum_{n=1}^{N-1} a_n \sin\left(\frac{\pi j n}{N}\right), \quad j=1,2,\ldots N-1 \tag{4}$$

where

$$a_n = \frac{2}{N} \sum_{j=1}^{N-1} f_j \sin\left(\frac{\pi j n}{N}\right), \quad n=1,2,\ldots N-1 \tag{5}$$

The derivatives of f at the discrete nodes θ_j can be approximated by

$$\frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta_j) = \sum_{n=1}^{N-1} n a_n \cos\left(\frac{\pi j n}{N}\right), \quad j=0,1,\ldots N \tag{6}$$

In the pseudospectral method, the derivatives of the discrete function are computed by Eq. (6) based on the spectral coefficients of Eq. (5). Derivatives of even periodic functions are approximated similarly. To apply this spectral technique in the circumferential direction, the term $\delta_\eta \hat{F}^n$ in the residue terms of Eq. (3) is replaced by the spectral approximations to the components of $(\partial \hat{F}/\partial \eta)^n$. The approximations to the η derivatives in the iteration operator on the left-hand side of Eq. (1) are retained in the finite difference form to preserve the tridiagonal block structure of the equations to be solved in the factored scheme. This amounts to further approximating a spectral approximation matrix, which is nonsparse, by a tridiagonal block matrix for iterative purposes. This does not degrade the convergence process of the iteration and the converged solution retains spectral accuracy in the circumferential direction.

High-order accurate spectral methods tend to exhibit nonlinear instabilities in convection-dominated flows and Gottlieb et al.[3] have shown that they can be stabilized by damping high-frequency waves. A low-pass filter has been applied to the spectral approximations by multiplying the spectral coefficients a_n of Eq. (5) by a filter function of the type used in Ref. 3.

Results

Figure 1 shows the pressure distributions on the leeside of a hemisphere cylinder computed by using both the spectral and fourth-order difference methods and the experimental data of Hsieh.[4] The flow Mach number is 1.2, its angle of attack is 19 deg, and it is assumed to be laminar. The mesh used ($48 \times 18 \times 30$) has 48 nodes along the body and there are 18 nodes in the circumferential direction so that half the circumference is divided into 15 intervals, or the circumferential nodes are 12 deg apart from each other. Also there are 30 nodes in the normal direction which are stretched out in such a way that approximately 15 nodes are in the thin viscous layer and 15 are in the outer region. The computations indicate that the spectral method gives results which are in better agreement with the experimental data than the fourth-order scheme. Downstream on the cylinder there is a significant crossflow separation and a suitable turbulence model would have to be added to get results closer to the experiment.

Figure 2 shows similar results computed on a coarser mesh ($30 \times 12 \times 30$), in which the circumferential nodes are 20 deg apart. Here, also, the spectral results are in better agreement with the experimental data than the fourth-order accurate results. In fact, the spectral results on the coarse mesh are in better agreement with the data than the fourth-order accurate results on the fine mesh of Fig. 1, at least in the forebody region. Pressure distributions on the windward side computed by both methods agree with each other and have reasonable agreement with the experiment except for a slight improvement in accuracy by the spectral method in the nose region. Figure 3 shows the surface streak-line patterns computed by both the methods and they simulate the oil flow pattern on the surface of the cylinder as photographed by Hsieh[4] and shown in Fig. 4. The nose separation pattern and the crossflow separation in the forebody region seem to be predicted better with the spectral method.

The calculations have been performed on a Cray computer taking advantage of its vector processing capabilities. The spectral version of the code requires approximately 15% more computing time than the original code but yields about a factor of 2 more accuracy (or requires a factor of 2 less nodes in the circumferential direction) for the cases run. Typically, accuracy increases rapidly with fineness of the mesh beyond some level in spectral methods. Thus, it is estimated that at least a 40% increase in efficiency can be gained by the spectral method developed here using a relatively sparse grid in the circumferential direction.

Conclusions

A pseudospectral approximation in the circumferential direction has been applied to the solution of the Navier-Stokes equations in three dimensions and has been shown to yield significant improvement in accuracy over a fourth-order finite difference approximation.

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Reynolds Stresses for Unsteady Turbulent Flows

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Introduction

REYNOLDS-averaged equations are currently employed in the numerical analysis of turbulent flows. In these equations an "apparent stress" that requires empirical data (e.g., eddy viscosity) is introduced to close the system and permit calculation of the flowfield. A predicament arises when calculating *unsteady* turbulent flows in which the empirical apparent stress level is of concern. For example, if all of the apparent stress obtained experimentally over the whole frequency spectrum were used in the calculation, this value would be added to the stress generated by the numerical algorithm, causing overestimation of the turbulence. This error would produce excessive dissipation of the unsteady phenomenon with the potential attendant disappearance of the unsteady solution. It is the purpose of this Note to address the procedure required to evaluate the frequency range of the Reynolds stress to be used for unsteady turbulent flows.

Reynolds-Averaged Equations

To demonstrate the procedure, the Reynolds-averaged equations will be examined for two-dimensional, incompressible flow. Only the x-momentum equation needs to be examined for this purpose,

$$(\rho u)_t + (\rho u^2 - \sigma_{11})_x + (\rho uv - \tau)_y = 0 \tag{1}$$

where $\sigma_{11} = -p + 2\mu u_x$ and $\tau = \mu(u_y + v_x)$.

To define the mean and fluctuating quantities in the classical manner, let

$$u = \bar{u}(x,y,t) + u'(x,y,t) \tag{2}$$

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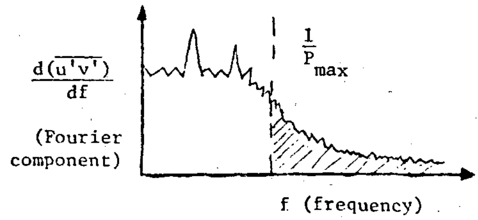

Fig. 1 Typical power spectral density plot.

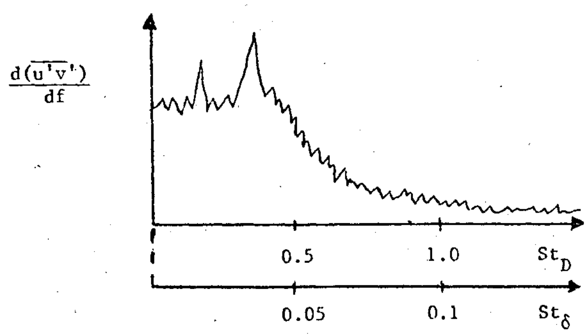

Fig. 2 Power spectral density vs Strouhal number.

$$v = \bar{v}(x,y,t) + v'(x,y,t) \tag{3}$$

$$p = \bar{p}(x,y,t) + p'(x,y,t) \tag{4}$$

where the fluctuating terms vanish over some time interval P, i.e.,

$$\frac{1}{P}\int_t^{t+P} u'\,\mathrm{d}t = 0 \tag{5}$$

The value of this P interval (as yet undefined) will be shown to be extremely important to the problem at hand. Inserting these values for the flow parameters into the x-momentum equation and integrating over the time interval P produces

$$(\rho\bar{u})_t + \left[\rho\bar{u}\bar{v} - \bar{\tau} + \frac{\rho}{P}\int_t^{t+P} u'v'\,\mathrm{d}t\right]_y + \left[\rho\bar{u}^2 - \bar{\sigma}_{11} + \frac{\rho}{P}\int_t^{t+P} u'^2\,\mathrm{d}t\right]_x = 0 \tag{6}$$

The two integrals are called the "apparent stress" terms. The second integral has little importance in most engineering problems since it is small compared to σ_{11}. However, the first integral is usually larger than the shear stress term τ and must be analyzed carefully. Defining

$$\overline{\overline{u'v'}} = \frac{1}{P}\int_t^{t+P} u'v'\,\mathrm{d}t \tag{7}$$

the Reynolds stress is therefore

$$\tau_t = -\rho\overline{\overline{u'v'}} \tag{8}$$

This average is taken over an interval P.

In the numerical calculation of unsteady turbulent flow all disturbances are resolved to within $\Delta s(\sqrt{\Delta x^2 + \Delta y^2})$ in space and Δt in time. No disturbance waves inside these increments can be resolved. This statement explains how the Reynolds-averaged equations are utilized, that is, the interval P over which the integration is taken for the averaging process must be related to a time step. Thus, $P = \Delta t$. Note that con-